Use the sum key to analyze rate of convergence.

A problem in Mathematics Magazine, December, 2012 asks us to show the value of this sum is $3\pi/4$ and to find an error estimate for the partial sums.

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right)$$

The published solution says the error when when adding just the first k terms is less than $2/k$.

Let's crunch numbers to find a better error estimate.

f1: 1, func, x^2, 1/x, 2, *, tan^{-1}

That defines function f1(n) to compute the n^{th} term in the sum. Now do a partial sum of the series.

40, sci, 7, func, 1, enter, 1e5, enter, 1, enter, 1, sum

This gives a sum, S, of $k = 100,000$ terms of the series. Use scientific notation to display results, since the numbers will get small.

S = 2.356174490292344595513649264124960509338e+0

If we didn't know the limiting value of the sum, we might try dividing by π, giving

7.499936338341072067016529363045201014493e−1

That makes us think S could be $0.75\,\pi$. We will guess that

$$S = \sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) = \frac{3\pi}{4} + \frac{c_1}{k} + \frac{c_2}{k^2} + \frac{c_3}{k^3} + \cdots$$

To try to guess the c_1 coefficient, compute $S - 3\pi/4$:

−1.999990000033333333327333466665380952381e−5

$-2/k$ would be −2.0e−5 here, so we guess $c_1 = -2$.

Next, to estimate c_2, look at $S - 3\pi/4$ + 2e−5

9.9999666666666672666533334619047618858733e−11

Guess $c_2 = 1$, compute $S - 3\pi/4$ + 2e−5 − 1e−10

−3.333333332733346666538095238114126664130e−16

Guess $c_3 = -1/3$ and add 1/3e+15 to that

5.999866667952380952192066692035333333333e−26

That last term dropped by about 1e−10 = $1/k^2$.

Try $c_4 = 0$ and $c_5 = 3/5$, and subtract 3/5e25

−1.333320476190478079333079646666666666667e−30

Guess $c_6 = -4/3$ and add 4/3e+30

1.285714285525400025368666666666666666667e−35

Guess $c_7 = 9/7$ and subtract 9/7e35

−1.888856889170476190476190476190479523810e−45

Guess c_8 is 0, since there was another 1e−10 drop.

Stopping here, we think that we have an approximation accurate to $O(1/k^9)$.

$$\sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) = \frac{3\pi}{4} - \frac{2}{k} + \frac{1}{k^2} - \frac{1}{3k^3} + \frac{3}{5k^5} - \frac{4}{3k^6} + \frac{9}{7k^7} + \cdots$$

As a check, compute the sum to $k = 23{,}456$ terms and see how close this formula comes. If the error is really $O(1/k^9)$, we would expect the difference from the true sum and the formula to be about $1/(23,456)^9$ = 5e−40 in magnitude.

$$\sum_{n=1}^{23{,}456} \arctan\left(\frac{2}{n^2}\right) = 2.3561092259798794253013541442481989674 69\text{e+0}$$

$$\frac{3\pi}{4} - \frac{2}{k} + \frac{1}{k^2} - \frac{1}{3k^3} + \frac{3}{5k^5} - \frac{4}{3k^6} + \frac{9}{7k^7} = 2.356109225979879425301354144248198967469\text{e+0}$$

Subtracting the formula from the sum gives −8.8e−40, which agrees fairly well with our prediction.

Making the same test with $k = 10$ gives (sum − formula) = −1.6e−9, which is also in good agreement with $O(1/k^9)$ accuracy.

To summarize, we have generated an accurate approximation for the partial sum as a function of k:

$$f(k) = \sum_{n=1}^{k} \arctan\left(\frac{2}{n^2}\right) \approx \frac{3\pi}{4} - \frac{2}{k} + \frac{1}{k^2} - \frac{1}{3k^3} + \frac{3}{5k^5} - \frac{4}{3k^6} + \frac{9}{7k^7}$$

$1/60^9$ is about 10^{-16}, so for any k bigger than 60, this formula should quickly give about full double precision accuracy (i.e., 16 significant digits for 64-bit double precision), without having to add k terms of the sum.

If we need to compute $f(k)$ for any positive k, we can just directly add the k terms of the arctan series if $k < 60$, or use the formula if $k \geq 60$. That way we can get an approximation for the partial sum that is both accurate and fast for any value of k.

The 50-digit precision of Calc-50 was needed in order to get this formula. Doing the calculations above for the c's while carrying only 16 digits would not have found any coefficients past c_2.