Arctangent Sum

Use the sum key to analyze rate of convergence.

A problem in Mathematics Magazine, December, 2011 asks us to show the value of this sum is $3\pi/4$ and to find an error estimate for the partial sums.

$$S = \sum_{k=1}^{\infty} \arctan\left(\frac{2}{k^2}\right)$$

The published solution says the error when when adding just the first n terms is less than $2/n$.

Let's crunch numbers to find a better error estimate.

f1: 1, func, x^2, 1/x, 2, $*$, tan^{-1}

That defines function f1(k) to compute the k^{th} term in the sum. Use the sum key to evaluate this infinite series.

40, fix, 7, func, 1, enter, e9999, enter, 1, enter, 1, sum

$S = 2.3561944901923449288469825374596271631479$

Even if we didn't know the value of this series, the presence of a trigonometric function might prompt us to divide by π:

π, /   0.7500000000000000000000000000000000000000

So the sum we got does seem to be $3\pi/4$.

To find an approximation for the error when adding just the first n terms of the series, get a partial sum,

$$S_n = \sum_{k=1}^{n} \arctan\left(\frac{2}{k^2}\right)$$

We will guess that

$$S_n = \sum_{k=1}^{n} \arctan\left(\frac{2}{k^2}\right) = \frac{3\pi}{4} + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots$$

In order to find these coefficients, we need to use a fairly large value of n. Picking n to be a power of 10 will also help us to recognize the values. Try $n = 10^6$.

40, fix, 7, func, 1, enter, e6, enter, 1, enter, 1, sum

$S_n = 2.3561924901933449285136492041268938284812$

Switch to scientific notation to display results, since the numbers will get small. To try to find the c_1 coefficient, subtract $3\pi/4$ from this:

40, sci, 3, π, $*$, 4, /, $-$

$$-1.9999990000003333333333332733334666665381\text{e}{-6}$$

$-2/n$ would be $-2.0\text{e}{-6}$ here, so it looks like $c_1 = -2$.
Next, to estimate c_2, look at $(S_n - 3\pi/4) + 2\text{e}{-6}$
by adding $2\text{e}{-6}$ to the previous result.

$$9.9999966666666666672666653333346190476190\text{e}{-13}$$

Using $c_2 = 1$ would make $c_2/n^2 = 1\text{e}{-12}$, so next
subtract $1\text{e}{-12}$ to get $(S - 3\pi/4 + 2\text{e}{-6}) - 1\text{e}{-12}$

$$-3.333333333327333346666653809523809542700\text{e}{-19}$$

Let $c_3 = -1/3$ and add $1/3\text{e}18$

$$5.999986666667952380952379063333333333333\text{e}{-31}$$

That last term dropped by about $1\text{e}{-12} = 1/n^2$.
Try $c_4 = 0$ and $c_5 = 3/5$, and subtract $3/5\text{e}30$

$$-1.333332047619047620936666666666666666667\text{e}{-36}$$

Use $c_6 = -4/3$ and add $4/3\text{e}36$

$$1.285714285712396666666666666666666663333\text{e}{-42}$$

With $c_7 = 9/7$, subtract $9/7\text{e}42$

$$-1.889047619047619047619050952380952380952\text{ee}{-54}$$

Guess c_8 is 0, since there was another $1\text{e}{-12}$ drop.

Stopping here, we think that we have an approximation accurate to $O(1/n^9)$.

$$\sum_{k=1}^{n} \arctan\left(\frac{2}{k^2}\right) \approx \frac{3\pi}{4} - \frac{2}{n} + \frac{1}{n^2} - \frac{1}{3n^3} + \frac{3}{5n^5} - \frac{4}{3n^6} + \frac{9}{7n^7}$$

Define function f2(n) to be this approximation.

$$\mathrm{f2}(n) = \frac{3\pi}{4} - \frac{2}{n} + \frac{1}{n^2} - \frac{1}{3n^3} + \frac{3}{5n^5} - \frac{4}{3n^6} + \frac{9}{7n^7}$$

It is a polynomial in terms of powers of $t = 1/n$, so we can use Horner's rule to evaluate it instead of using the $\mathrm{y^x}$ key for each separate power.

$$\mathrm{f2}(n) = \left(\left(\left(\left(\left(\left(\frac{9}{7}t - \frac{4}{3}\right)t + \frac{3}{5}\right)t + 0\right)t - \frac{1}{3}\right)t + 1\right)t - 2\right)t + \frac{3\pi}{4}$$

Four enter commands fill the stack with the t value (one will be deleted when the next item is a number instead of a command, leaving three in the program). Then whenever the stack drops during the chain of operations after that, a copy of t remains at the top of the stack, so as many copies of t as we need will be available. The "$*$, $*$," in the f2 definition reflects the fact that the "+ 0" in Horner's rule from the missing $1/n^4$ term can be skipped.

f2: 1, x↔y, /, enter, enter, enter, enter, 9, enter, 7, /, ∗, 4, enter, 3, /, −, ∗, 3, enter, 5, /, +, ∗, ∗, 1, enter, 3, /, −, ∗, 1, +, ∗, 2, −, ∗, 3, π, ∗, 4, /, +

As a check, compute the sum to $n = 23{,}456$ terms and see how close this formula comes. If the error is really $O(1/n^9)$, we would expect that (formula − true sum) would be about $1/(23{,}456)^9 = 5\text{e}{-40}$ in magnitude.

7, func, 1, enter, 23456, enter, 1, enter, 1, sum 2.356109225979879425301354144248198967469e+0

2, sto, 10, sci, 23456, enter, 2, $\mathrm{f_n}$, 2, rcl, − 8.787988264e−40

Subtracting the sum from the formula gave 8.8e−40, which agrees fairly well with our prediction.

Checking with $n = 10$ gives (formula − sum) = 1.6e−9, also in good agreement with $O(1/n^9)$ accuracy.

To summarize, we have generated an accurate approximation for the partial sum as a function of n:

$$f(n) = \sum_{k=1}^{n} \arctan\left(\frac{2}{k^2}\right) \approx \frac{3\pi}{4} - \frac{2}{n} + \frac{1}{n^2} - \frac{1}{3n^3} + \frac{3}{5n^5} - \frac{4}{3n^6} + \frac{9}{7n^7}$$

$1/60^9$ is about 10^{-16}, so for any n bigger than 60, this formula should quickly give about full double precision accuracy (i.e., 16 significant digits for 64-bit double precision), without having to add n terms of the sum.

If we need to compute $f(n)$ for any positive n, we can just directly add the n terms of the arctan series if $n < 60$, or use the formula if $n \geq 60$. That way we can get an approximation for the partial sum that is both accurate and fast for any value of n.

The 50-digit precision of Calc-50 was needed in order to get this formula. Doing the calculations above for the c's while carrying only 16 digits would not have found any coefficients past c_2.