

Derive a Gauss quadrature integration rule.

There are several kinds of Gauss rules. We will consider the Gauss-Legendre rule.

To find a Gauss formula we will need a function that evaluates Legendre polynomials. This derivation uses combinations, derivatives, sums, graphing, and equation solving.

The Gauss quadrature formula is
$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

Each different value for n gives a different integration rule. There are n function values to evaluate and we need to find the $2n$ constants w_1, w_2, \dots, w_n and x_1, x_2, \dots, x_n .

These constants depend on the n^{th} degree Legendre polynomial:

$$P_n(x) = 2^{-n} \sum_{k=0}^n \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k$$

The x_i values are the roots of P_n , and $w_i = 2/((1-x_i^2)(P_n'(x_i))^2)$

First, define a function to evaluate $P_n(x)$.

f1: 1, sto, 0, rcl, 1, rcl, cmb, x², 2, rcl, 1, -, 0, rcl, 1, rcl, -, y^x, *, 2, rcl, 1, +, 1, rcl, y^x, *

f2: 2, sto, 0, enter, 0, rcl, 1, enter, 1, sum, 2, enter, 0, rcl, y^x, /

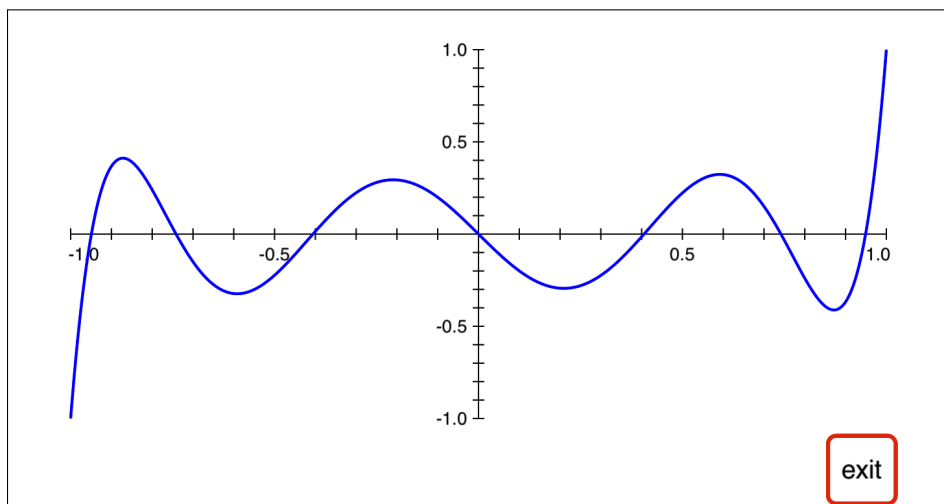
n is stored in register 0 before using f2, x is in register 2, k is in register 1

f1 saves k in register 1, then computes the k^{th} term in the Legendre sum

f2 saves x in register 2, then sums f1 for $k = 0$ to n by steps of 1 in order to compute $P_n(x)$

For a specific example, to derive the 7-point rule, let $n = 7$ and draw the graph of $P_7(x)$ to get a first guess for the locations of the roots:

7, enter, 0, sto, -1, enter, 1, enter, 2, plot



There are 7 roots, near $x = 0.0, \pm 0.4, \pm 0.7, \pm 0.9$.

Use solve to find the roots (x_i in the Gauss formula), and save them in registers 11, 12, ..., 17

-.9, enter, 2, solv, 11, sto	$x_1 = -0.949107912342758524526189684048$
-.7, enter, 2, solv, 12, sto	$x_2 = -0.741531185599394439863864773281$
-.4, enter, 2, solv, 13, sto	$x_3 = -0.405845151377397166906606412077$
0, enter, 2, solv, 14, sto	$x_4 = 0.00000000000000000000000000000000$
.4, enter, 2, solv, 15, sto	$x_5 = 0.405845151377397166906606412077$
.7, enter, 2, solv, 16, sto	$x_6 = 0.741531185599394439863864773281$
.9, enter, 2, solv, 17, sto	$x_7 = 0.949107912342758524526189684048$

Next use the f' key to find $P_n'(x_i)$ and compute the w 's

f3: 10, +, 3, sto, rcl, 2, f', x², 3, rcl, rcl, x², chs, 1, +, *, 1/x, 2, *

f3 adds 10 to i to get the register number where x_i is saved, then recalls x_i and applies the f' function to f2, then squares the result.

3, rcl gets the register number back for x_i , then the second rcl gets the x_i value, squares that, computes $(1 - x_i^2)$, multiplies by $P_n'(x_i)^2$, inverts, and multiplies by 2.

Compute the w_i values and save them in registers 21, 22, ..., 27

1, enter, 3, f _n , 21, sto	$w_1 = 0.129484966168869693270611432679$
2, enter, 3, f _n , 22, sto	$w_2 = 0.279705391489276667901467771424$
3, enter, 3, f _n , 23, sto	$w_3 = 0.381830050505118944950369775489$
4, enter, 3, f _n , 24, sto	$w_4 = 0.417959183673469387755102040816$
5, enter, 3, f _n , 25, sto	$w_5 = 0.381830050505118944950369775489$
6, enter, 3, f _n , 26, sto	$w_6 = 0.279705391489276667901467771424$
7, enter, 3, f _n , 27, sto	$w_7 = 0.129484966168869693270611432679$

For a test of these values, use them to approximate $\int_{-1}^1 e^x dx$

f4: e^x

f5: 5, sto, 10, +, rcl, 4, f_n, 5, rcl, 20, +, rcl, *

f4 is the function to be integrated

f5 saves i in register 5, then computes $w_i f4(x_i)$

Now the Gauss quadrature approximation to the integral is $w_1 f(x_1) + w_2 f(x_2) + \dots + w_7 f(x_7)$

1, enter, 7, enter, 1, enter, 5, sum 2.350402387287600752998338042877

Check with the integrate key

-1, enter, 1, enter, 4, \int_a^b 2.350402387287602913764763701191

The $n = 7$ version of Gauss quadrature gives 15 correct significant digits.