Derive a Guass quadrature rule for approximating definite integrals.

There are several kinds of Gauss rules. We will consider the Gauss-Legendre rule.

To find a Gauss-Legendre formula we will need a function that evaluates Legendre polynomials. This derivation uses combinations, derivatives, sums, graphing, and equation solving.

The Guass quadrature formula is $$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i\, f(x_i)$$

Each different value for n gives a different integration rule. There are n function values to evaluate, and to define the rule we need to find the $2n$ constants $w_1, w_2, \ldots, w_n$ and $x_1, x_2, \ldots, x_n$.

These constants depend on the n^{th} degree Legendre polynomial:

$$P_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}\,(x+1)^k$$

The x_i values are the roots of P_n, and $w_i = 2/((1-x_i^2)(P_n{}'(x_i))^2)$

First, define a function to evaluate $P_n(x)$.

f1: 3, func, 1, sto, 0, rcl, 1, rcl, cmb,
1, func, x^2, 2, rcl, 1, $-$, 0, rcl, 1, rcl, $-$, $\mathrm{y^x}$, *, 2, rcl, 1, +, 1, rcl, $\mathrm{y^x}$, *

f2: 7, func, 2, sto, 0, enter, 0, rcl, 1, enter, 1, sum, 1, func, 2, enter, 0, rcl, $\mathrm{y^x}$, /

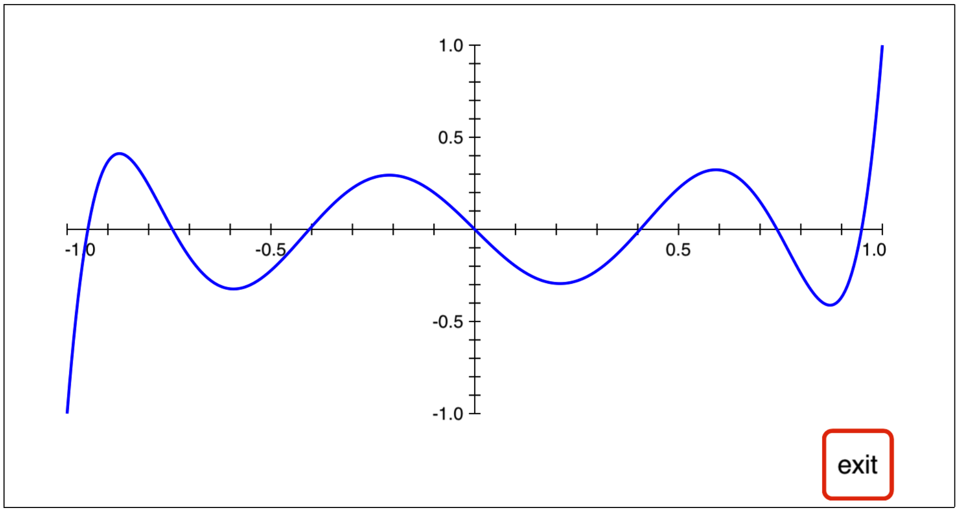
n is stored in register 0 before using f2, x is in register 2, k is in register 1.

f1 saves k in register 1, then computes the k^{th} term in the Legendre sum.

f2 saves x in register 2, then sums f1 for $k = 0$ to n by steps of 1 in order to compute $P_n(x)$.

Derive the 7-point rule. Set $n = 7$ and plot $P_7(x)$ to get a first guess for the locations of the roots:

7, func, 7, enter, 0, sto, -1, enter, 1, enter, 2, plot



There are 7 roots, near $x = 0.0,\ \pm 0.4,\ \pm 0.7,\ \pm 0.9$.

Use solve to find the roots (x_i in the Gauss formula), and save them in registers $11, 12, \ldots, 17$

6, func, -.9, enter, 2, solv, 11, sto	$x_1 = -0.9491079123427585245261896840\,48$
-.7, enter, 2, solv, 12, sto	$x_2 = -0.741531185599394439863864773281$
-.4, enter, 2, solv, 13, sto	$x_3 = -0.405845151377397166906606412077$
0, enter, 2, solv, 14, sto	$x_4 = \ 0.000000000000000000000000000000$
.4, enter, 2, solv, 15, sto	$x_5 = \ 0.405845151377397166906606412077$
.7, enter, 2, solv, 16, sto	$x_6 = \ 0.741531185599394439863864773281$
.9, enter, 2, solv, 17, sto	$x_7 = \ 0.949107912342758524526189684048$

Next use the f′ key to find ${P_n}'(x_i)$ and compute the w's

f3: 10, +, 3, sto, rcl, 2, f′, 1, func, x^2, 3, rcl, rcl, x^2, chs, 1, +, *, 1/x, 2, *

f3 adds 10 to i to get the register number where x_i is saved, then recalls x_i and applies the f′ function to f2, then squares the result.

3, rcl gets the register number back for x_i, then the second rcl gets the x_i value, squares that, computes $(1 - x_i^2)$, multiplies by ${P_n}'(x_i))^2$, inverts, and multiplies by 2.

Compute the w_i values and save them in registers $21, 22, \ldots, 27$

1, enter, 3, f$_\mathrm{n}$, 21, sto	$w_1 = 0.129484966168869693270611432679$
2, enter, 3, f$_\mathrm{n}$, 22, sto	$w_2 = 0.279705391489276667901467771424$
3, enter, 3, f$_\mathrm{n}$, 23, sto	$w_3 = 0.381830050505118944950369775489$
4, enter, 3, f$_\mathrm{n}$, 24, sto	$w_4 = 0.417959183673469387755102040816$
5, enter, 3, f$_\mathrm{n}$, 25, sto	$w_5 = 0.381830050505118944950369775489$
6, enter, 3, f$_\mathrm{n}$, 26, sto	$w_6 = 0.279705391489276667901467771424$
7, enter, 3, f$_\mathrm{n}$, 27, sto	$w_7 = 0.129484966168869693270611432679$

For a test of these values, use them to approximate $\int_{-1}^{1}\ e^x\ dx$

f4: 1, func, e$^\mathrm{x}$

f5: 5, sto, 10, +, rcl, 4, f$_\mathrm{n}$, 5, rcl, 20, +, rcl, *

f4 is the function to be integrated

f5 saves i in register 5, then computes w_i f4(x_i)

Now the Gauss quadrature approximation to the integral is $w_1\, f(x_1) + w_2\, f(x_2) + \cdots + w_7\, f(x_7)$

7, func, 1, enter, 7, enter, 1, enter, 5, sum   2.350402387287600752998338042877

Check with the integrate key

6, func, −1, enter, 1, enter, 4, $\int_a^b$ 2.350402387287602913764763701191

The $n = 7$ version of Gauss quadrature gives 15 correct significant digits.