Oscillating Integrals

Integrals of functions that are highly or infinitely oscillatory often cannot be handled by standard numerical integration formulas. Here is a difficult numerical integral, from the "Hundred-digit Challenge" published in 2002 by Nick Trefethen (in Wikipedia, search for "Hundred-digit Challenge").

$$\int_0^1 \frac{1}{x} \cos\left(\frac{\ln(x)}{x}\right)\, dx$$

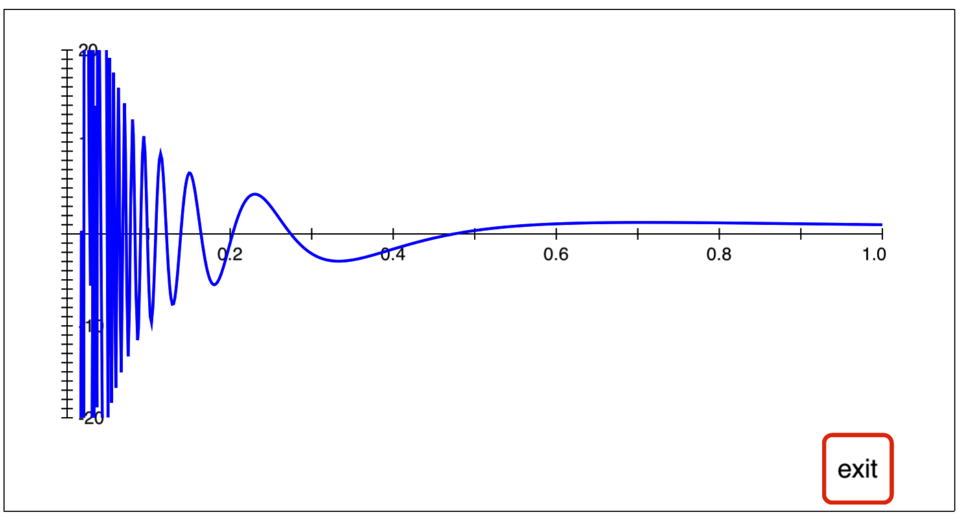
The Hundred-digit Challenge was a collection of ten hard problems, where in each case the goal was to find 10 significant digits for the answer. We will see how to get 50 digits.

The function to be integrated oscillates infinitely often with higher and higher amplitude as x approaches 0. Define f1 to be this function and look at its graph.

f1: 1, sto, 1, func, ln, 1, rcl, /, cos, 1, rcl, /

Set ymax and ymin to cut the graph off above $y = 20$ and below $y = -20$.

7, func, 20, ymax, −20, ymin, 0, enter, 1, enter, 1, plot



Dirk Laurie calls this graph "appalling" in the chapter on this problem from the "SIAM 100-digit Challenge" book (SIAM stands for the Society for Industrial and Applied Mathematics). The oscillations become so violent near zero that Calc-50's plot function can't quite keep up. Try the integrate key.

0, enter, 1, enter, 1, $\int_a^b$

The integrate function agrees that the function is appalling, returning "unknown".

So it is on to Plan B. Does the integral even exist? We can make a change of variables to spread the wobbles out — that might help us see how it behaves. Try making the substitution $u = 1/x$. That will send the singularity at $x = 0$ off to infinity and spread the oscillations out.

$$x = u^{-1}$$

$$dx = -u^{-2}\ du$$

$$x \to 0 \quad \Rightarrow \quad u \to \infty \qquad\qquad x = 1 \quad \Rightarrow \quad u = 1$$

This gives an equivalent integral

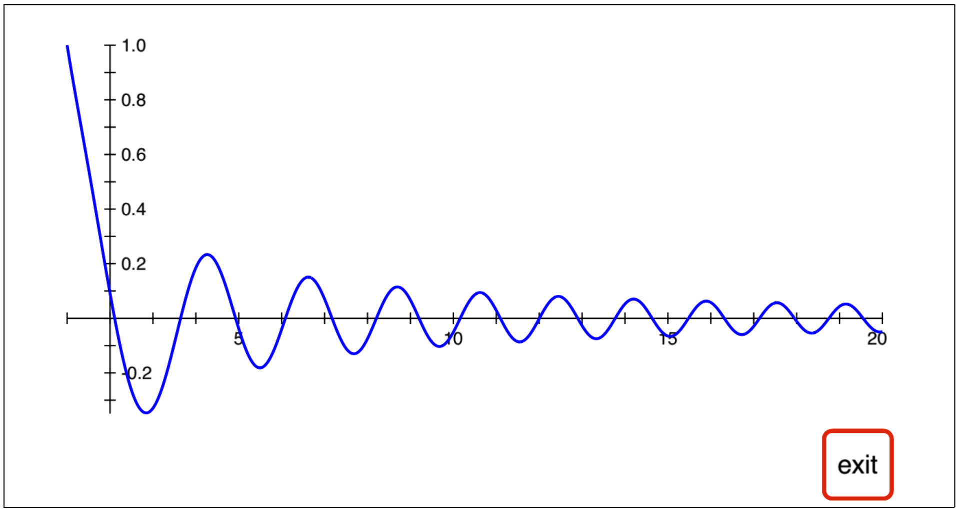
$$\int_{\infty}^{1} u\ \cos\left(u\ \ln(u^{-1})\right)(-u^{-2})\ du \ = \ \int_{1}^{\infty} \frac{\cos\left(u\ \ln(u^{-1})\right)}{u}\ du \ = \ \int_{1}^{\infty} \frac{\cos\left(u\ \ln(u)\right)}{u}\ du$$

Define f2 to be this second function and look at its graph.

f2: 2, sto, 1, func, ln, 2, rcl, ∗, cos, 2, rcl, /

Set ymax and ymin back to automatic for this plot.

7, func, 0, ymax, ymin, 1, enter, 20, enter, 2, plot



That looks a little better, but the integrate key still can't do this integral from 1 to infinity. However, this second function is well-behaved and easy to integrate between successive roots. A series of integrals from one root to the next would give the terms in an infinite series, and the sum of that series is the value of the integral. The size of the terms in this series is decreasing and the signs are alternating, so the series converges. That means the original integral does exist.

We will compute these integrals and then use extrapolation to estimate their sum. Because the individual terms are each integrals, they are too slow and too complicated for the sum key to automatically evaluate the infinite series in the way it could for the examples on the "Infinite sums" page.

The roots of $\cos(x)$ are $\pi/2$, $3\pi/2$, $5\pi/2$, $\cdots$, $(2k-1)\pi/2$, $\cdots$, so our second function will be zero when

$$x\ \ln(x) = (2k-1)\pi/2.$$

The function $x\ \ln(x)$ is positive and strictly increasing for $x > 1$, so this equation has only one root for a given k, and the solve key can quickly find it. The roots slowly get closer together as x increases, and the first one ($k = 1$) is close to 2. Looking at the graph makes it seem that $x = k$ is a decent starting point for the solve key's root search for root number k.

Function f3 will be $x\ \ln(x) - (2k-1)\pi/2$, with x the input and k having been stored in register 10.

f3: 3, sto, 1, func, ln, 3, rcl, $*$, 10, rcl, 2, $*$, 1, $-$, π, $*$, 2, /, $-$

Call the k^{th} root $r(k)$, with the special case $r(0) = 1$, and integrate f2 from $r(k)$ to $r(k+1)$ for $k = 0, 1, 2, \ldots n$, to get the terms in our sum of integrals.

f4(k) computes $r(k)$ by selecting f5 to give 1 when $k = 0$, or f6 to use the solve function for $r(k)$ when $k > 0$.

f4: 7, func, 0, enter, 0, enter, 5, sel

f5: 1

f6: 10, sto, 3, solv

f7(k) will find the integral of f2 from $r(k)$ to $r(k+1)$.

f7: 7, sto, 4, f_n, 17, sto, 7, rcl, 1, +, 4, f_n, 17, rcl, x↔y, 2, $\int_a^b$

Check the first few roots.

30, fix, 0, enter, 4, f_n	1.000000000000000000000000000000
1, enter, 4, f_n	2.107299476814416527742351541493
2, enter, 4, f_n	3.644173671645632136171294253899

Those seem to agree with the graph. Check the first few integrals.

0, enter, 7, f_n	0.549449923651782130358036700587
1, enter, 7, f_n	−0.340072436812882195282463574316
2, enter, 7, f_n	0.190555879387673920042091339668

Those also look ok. Extrapolate the sum of the first 100 integrals. See the "Extrapolation" page for more details on the extr key.

40, fix, 7, func, 1, enter, 0, enter, 99, enter, 7, extr		0.3233674316777787613993700879521704466510
x↔y, 10, sci	(estimated relative error)	1.000000000e−55

As a check on this result, a high-precision value is given at https://oeis.org/A117231, confirming that we have over 50 digits correct.