Euler-Maclaurin Formula

Here is a slowly convergent series: $$S = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{k})}{k}$$

Several Calc-50 example pages mention that the sum key on screen 7 sometimes uses the Euler-Maclaurin formula to get high-accuracy approximations for sums.

The sum key cannot handle this series, because the sum key needs the signs of the terms to be either all the same or strictly alternating. This series has groups of several positive terms followed by several negative terms.

If we can compute the Euler-Maclaurin approximation "by hand", we might be able to get around this problem. Here is one form of the Euler-Maclaurin formula:

$$\sum_{k=a}^{b} f(k) \;\sim\; \int_a^b f(x)\,dx \;+\; \frac{f(a)+f(b)}{2} \;+\; \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(b) - f^{(2k-1)}(a)\Big)$$

This says the expression on the right is an asymptotic approximation to the sum. The sum on the right involving Bernoulli numbers and derivatives of f may not converge, but for a fixed number of terms in that sum, as a gets larger the accuracy improves for the right side as an approximation to the sum on the left.

Since $f(x)$ oscillates between positive and negative values infinitely often from a to infinity, the Calc-50 integration function cannot evaluate this integral automatically. Here we can use extrapolation to compute the integral, as in the "Oscillating Integrals" page.

For the S sum we have $f(x) = \sin(\sqrt{x})/x$, and in the asymptotic formula we want $b \to \infty$, making $f(b)$ and the derivatives $f^{(2k-1)}(b)$ negligible. This gives

$$\sum_{k=a}^{\infty} \frac{\sin(\sqrt{k})}{k} \;\sim\; \int_a^{\infty} \frac{\sin(\sqrt{x})}{x}\,dx \;+\; \frac{\sin(\sqrt{a})}{2\,a} \;-\; \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\, f^{(2k-1)}(a)$$

We need for a to be fairly large in order to get the asymptotic approximation to be accurate enough, so we will choose $a = 100{,}000$. This means the Euler-Maclaurin approximation will estimate the S sum from 100,000 to infinity, and we will explicitly add the first 99,999 terms.

$$S = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{k})}{k} \;\approx\; \sum_{k=1}^{99{,}999} \frac{\sin(\sqrt{k})}{k} \;+\; \int_{100{,}000}^{\infty} \frac{\sin(\sqrt{x})}{x}\,dx \;+\; \frac{\sin(\sqrt{100{,}000})}{200{,}000} \;-\; \sum_{k=1}^{10} \frac{B_{2k}}{(2k)!}\, f^{(2k-1)}(100{,}000)$$

The first and third terms are straightforward, and extrapolation should get the integral. The final sum will be limited to $k \le 10$ since Calc-50 provides derivatives only up to 20^{th} order. Checking these 10 terms gives $-6.36\text{e}{-10}$, $-2.76\text{e}{-17}$, ..., $-1.78\text{e}{-67}$, $-1.20\text{e}{-74}$.

This suggests that even after restricting the last sum to 10 terms we should still be able to get over 50 digits correct for S.

(a) First and third terms

Define f1 to be $\sin(\sqrt{x})/x$.

f1: 1, sto, 1, func, $\sqrt{\text{x}}$, sin, 1, rcl, /

Compute the sum of the first 99,999 terms and add $\sin(\sqrt{100{,}000})/200{,}000$. Store the result in register 11.

30, fix, 7, func, 1, enter, 99999, enter, 1, enter, 1, sum, 1, func, e5, $\sqrt{\text{x}}$, sin, 2e5, /, +, 11, sto

1.7186735647953557193145058413 66

(b) Integral

As in the "Oscillating Integral" page, make a sum by adding the integrals from one root of the function to the next, then extrapolate. The n^{th} positive root is when $\sqrt{x} = n\pi$, so $x = (n\pi)^2$. The first root larger than 100,000 is $x = (101\pi)^2 = 100{,}679.834\ldots$ Define f2(k) to integrate f1 from $((100+k)\pi)^2$ to $((100+k+1)\pi)^2$.

f2: 1, func, 2, sto, 100, +, π, *, x^2, 2, rcl, 101, +, π, *, x^2, 6, func, 1, $\int_a^b$

Extrapolate using the first 100 terms of the sum of f2 to estimate the integral from $(101\pi)^2$ to infinity.

7, func, 1, enter, 1, enter, 100, enter, 2, extr, 12, sto −0.0063030408658786015992535270 28

Pressing x↔y to see the extrapolation key's estimated error shows 1.0e−55, so we should have 50 digits correct for that part of the integral. Get the rest of the integral term by integrating from 100,000 to $(101\pi)^2$. Add that to the part above and store the entire integral's value in register 12.

1, func, 100000, π, 101, *, x^2, 6, func, 1, $\int_a^b$, 12, rcl, +, 12, sto −0.0030017707225995057827863478 04

(c) Final sum

f3(k) computes term k in the sum that has Bernoulli numbers and derivatives.

f3: 2, *, 3, sto, 1, −, e5, x↔y, 1, f$^{(n)}$, 3, func, 3, rcl, B$_{\text{n}}$, *, 3, rcl, x!, /

Add the first 10 terms, change the sign since this sum is subtracted in the Euler-Maclaurin formula, and store the result in register 13.

7, func, 1, enter, 10, enter, 1, enter, 3, sum, chs, 13, sto 0.0000000006363647984635559990 60

Add registers 11, 12, and 13 to get an approximation for S.

11, rcl, 12, rcl, +, 13, rcl, + 1.7156717947091210119952754926 23